Mathematics: applications and interpretation HL

Timezone 1

To protect the integrity of the assessments, increasing use is being made of examination variants. By using variants of the same examination, students in one part of the world will not always be responding to the same examination content as students in other parts of the world. A rigorous process is applied to ensure that the content across all variants is comparable in terms of difficulty and syllabus coverage. In addition, measures are taken during the standardization and grade awarding processes to ensure that the final grade awarded to students is comparable.

Contents

Grade boundaries

Higher level overall

Grade:	1	2	3	4	5	6	7
Mark range:	0–12	13–25	26–36	37–50	51–62	63–73	74–100

Higher level internal assessment

Grade:	1	2	3	4	5	6	7
Mark range:	0–2	3–5	6–8	9–11	12–14	15–16	17–20

Higher level paper one

Grade:	1	2	3	4	5	6	7
Mark range:	0–14	15–26	27–35	36–47	48–57	58–68	69–110

Higher level paper two

Grade:	1	2	3	4	5	6	7
Mark range:	0–15	16–30	31–41	42–59	60–74	75–88	89–110

Higher level paper three

Grade:	1	2	3	4	5	6	7
Mark range:	0–7	8–14	15–21	22–28	29–35	36–41	42–55

Higher level internal assessment

The range and suitability of the work submitted

Although there were some very good explorations, overall, the standard was lower than might be expected this far through the cycle. There was often a problem with statistical explorations, due to the statistic tests applied not being valid, the conditions of the test e.g. normality, were often not checked and there were case of χ^2 tests being used on quantities rather than on frequencies. If doing a statistical investigation teachers need to ensure the funadamentals are well understood.

In terms of the range of explorations there were many that were unique, involving a problem designed by the student around something personal to them. These were usually the most interesting to read and were likely to score well. Against this there were some that looked at familiar topics such as finding the surface area of a bottle, or the cooling of a cup of coffee. Though these tasks might be new to the student it is difficult for the moderator to fully hear the student 'voice' if they are following such a well-worn route.

A small number of students still venture into complex and abstract mathematical concepts and models, including optimization problems characterized by numerous decision variables and constraints. These topics, while demonstrating ambition, often proved too sophisticated for a thorough exploration within the IA's scope, potentially hindering the clarity and depth of mathematical investigation.

The trend of choosing topics heavily reliant on other disciplines persisted, with significant portions of some IAs dedicated to explaining concepts from physics, biology, and computer science. The emphasis on these interdisciplinary areas sometimes overshadowed the mathematical analysis, leading to a diminished focus on mathematical principles and techniques.

There were clear instances in this session of students adopting a template for the production of their work. One instance of this, that was seen often by the moderators, was using a Voronoi diagram to help with university choice and then the TSP to find a route around these choices. Explorations that follow a template are unlikely to score well in several of the criteria, and if any exploration proves to be too similar to another or to a similar script posted online, the student will be at risk of malpractice. In addition, following such routes does not allow the student to fully benefit from doing an investigative piece of work and developing the associated skills, which is one of the main goals of the course.

There was very little explicit evidence of AI tool assistance in writing IAs. Schools are reminded that where used this needs to be carefully referenced.

Student performance against each criterion

Criterion A

Most students seem to be aware of the importance of the organization of a task. However, coherence in Criterion A should be highlighted as well. This includes the idea of the work being logically developed, but is also a measure of how easy it is for the reader to understand. For example, a mathematics IA containing lots of scientific terminology, sophisticated physics formulae or the minutiae of a computer game might be logically developed by the author, but not easy to follow for readers. Care must be taken to make the work easy to read, and this applies equally to the language and context as well as the manipulation of the mathematics.

A reminder once again that there is no need to describe in detail techniques or ideas that are on the syllabus or to show very basic calculations, such as how to find a mean. Doing this can make the work 'not concise'.

A table of contents, a plan or description of method is not required.

A discussion of limitations at the end is often not helpful. It is much better for the student to discuss these as the work progresses.

Criterion B

As always there was plenty of good mathematical communication with the usual errors seen, for example,

- Final answers should be appropriately rounded and include units
- Use subscripts where appropriate (X_1 not $X1$)
- Tables that go over two pages need new column headings
- A description of GDC use is not appropriate communication
- An absence of diagrams can make the work difficult to follow
- Diagrams should be created by the student where possible, and where not should be referenced and should not contain any superfluous information
- An equation editor should be used, and variables should be italicised

Criterion C

Though there were some excellent explorations seen, the modal score in this criterion during this session was C1. In order to display ‘significant’ or ‘outstanding’ personal engagement it is not enough for the student to just express an interest or even to collect data or research beyond the syllabus. They would normally need to engage in some aspect of problem solving.

The following extract from the guide should also be noted: **‘Personal engagement may be recognized in different ways. These include thinking independently or creatively, presenting mathematical ideas in their own way, exploring the topic from different perspectives, making and testing predictions.’**

This means that the level that can be awarded in criterion C is often limited by the choice of task. A routine task, even if it has been personalized (they have chosen their own vase to model), will not provide much opportunity for creative or independent thinking.

When developing their own ideas students can be rewarded in this criterion for original thinking, even if mathematically the work is not always valid.

Criterion D

Students’ reflections are often descriptive, simply stating the result of the test or qualitatively describing potential sources of error.

There is potential for meaningful reflection when thinking about the context of the exploration. For example, when outliers or influential points are identified it is important to reflect on why they are distanced from the other data points, and so whether or not they should be included. Meaningful reflection can also be seen in the construction of the method used to collect primary data for analysis and in testing the results of that analysis with different data.

Critical reflection will normally include reflection on the mathematical process (was the statistical method used a valid one, for example) or a mathematical analysis of errors.

Criterion E

Generally, work that comes only from the standard level section of the course will not be regarded as sophisticated **'unless the mathematics has been used in a complex way that is beyond what could reasonably be expected of an SL student'** (Course Guide, p 89). That said, complex mathematics is not necessarily sophisticated if it has not been used in a sophisticated way, for example using a complex method when there is a simpler method (that is within the syllabus) available to them. As an example, fitting a cubic curve by taking four points and using matrix techniques might demonstrate a good understanding of the techniques, but the work is not sophisticated unless the student can provide a reason why this method is an improvement on using the least squares regression curve.

Students need to be aware that incomplete explanations cannot indicate good understanding, particularly when explaining work beyond the syllabus. Care should be taken to explain the process clearly and in their own words.

It is not enough to just complete the techniques correctly to be assessed as demonstrating 'good' understanding, the techniques used need to be the correct ones for the task and also interpreted correctly. The writing should also contain an explanation of the mathematics used, and why it is applicable in their chosen situation. Some students, when modelling, were demonstrating lots of good mathematical techniques, for example finding maximum points using calculus, but in doing so were showing only limited knowledge regarding modelling.

Where the exploration contains a major error that indicates a lack of understanding on the part of the student, then it could be deemed to be 'only partially correct'. Small errors though can sometimes still be seen in work in that is otherwise considered to demonstrate 'good' understanding.

Recommendations and guidance for the teaching of future students

The goal of all IB Mathematics courses is to have the student able to explore mathematics independently and to develop the personal and mathematical skills required to overcome problems as they arise. This is a long process which should begin even before the start of the diploma program. Within the diploma program the hours allocated to this by the IB (the toolbox) should be fully utilized.

Students need to be reminded that it is not 'cheating' or 'unmathematical' to use software to perform calculations rather than to do them by hand. Indeed, not doing so in an exploration will leave more time for those skills, abstraction, analysis, evaluation and improvements that will lead to the award of the higher levels.

With regard to statistics, it is important students have a good understanding of which tests are available to them and when they should be used and what the underlying assumptions are. As in examinations, they need to state the hypotheses, the p-value and the conclusion. There is no need for any detailed description of the calculations.

Consideration needs to be given to the order in which topics are taught to ensure that the students have sufficient knowledge of the course to write a successful exploration.

Further comments

Teachers need to be fully aware of their roles in supporting students to find an appropriate topic for their exploration, in reading a first draft and including in their feedback an indication of the presence of errors, and in marking and annotating the script for submission.

There is still a high correlation between the schools whose teachers annotate their scripts and those whose marks are left unchanged by moderation.

Some schools are submitting an unmarked copy of the exploration and separately one with teacher comments. This is not helpful to the moderator whose role it is to confirm the teacher marks, so please just upload the annotated script.

Teachers should be aware of the existence of templates for the exploration and of the use of artificial intelligence tools. They need to be sure that the student's ideas are their own and they develop them themselves. If in doubt they should perform a viva voce. They should not allow the submission of work they cannot guarantee is the student's own.

Higher level paper one

General comments

Apart from question 4, the students performed reasonably well, in general, on the first half of the paper. The second half of the paper, however, proved to be difficult for many students, with two of the questions, listed below, attracting only a handful of complete solutions.

The areas of the programme and examination which appeared difficult for the students

- Most students are unable to use the Euler Method to find an approximate solution to a differential equation.
- Many students are unfamiliar with Voronoi diagrams.
- While students are generally confident in using their GDC to evaluate a definite integral, knowledge of indefinite integration is not so good in general.
- The manual calculation of the sum of squared residuals.
- Matrix multiplication.
- Phase portraits.
- Matrix diagonalization applied to Markov Chains.

The areas of the programme and examination in which students appeared well prepared

- Many students are confident using the finance software on their GDC.
- Use of the GDC to find correlation coefficients and regression lines.
- The calculation of the terms and sums of geometric series.

The strengths and weaknesses of the students in the treatment of individual questions

Question 1

Many students gave a correct solution to part (a), either by using the finance app on their GDC or by using the compound interest formula. The most common error was a failure to assign opposite signs to PV and PMT. Part (b) was found difficult by many students, possibly not knowing exactly what an annuity is. Again, a common error was a failure to assign opposite signs to PV and PMT. Some students tried to use an annuity formula, but this was rarely successful.

Question 2

Part (a) was well answered by most students although some students lost marks by giving the answer $y = 10.6x + 43.9$ in part (ii) instead of a relationship between m and t. In Part (b), many students gave the incorrect answer $10.6 \times 1.5 + 43.9$. A variety of reasons were given in part (c), some correct, some incorrect.

Question 3

Part (a) was the best answered question on the paper with most students able to invert the relationship to give an expression for C in terms of F and to go on to convert 77°F to C°. In part (b), however, whereas most students gave the correct mean Fahrenheit temperature, the majority of students gave the incorrect answer $1.8 \times 9 + 32$ for the standard deviation, not realizing that additive constants do not affect the standard deviation.

Question 4

Very few students gave a complete solution to this question. Some students copied the general iterative formula from the formula booklet but were then unable to apply this to the given equation. Some students managed to show correctly that $y(0.1) = 1$.

Question 5

Most students gave correct answers to (a) and (b) although some students lost a mark in part (b) by giving the answer as $\left(\frac{48}{7}, \frac{17}{7}\right)$ instead of correct to four significant figures as required. Many students did not attempt (c), suggesting perhaps that some schools failed to teach the theory of Voronoi diagrams. A common error was to join up the three points A, B, C or even A, B, C, V.

Question 6

Most students realized that this question was based on geometric sequences and most solved part (a) correctly. In part (b), most students realized that they had to sum terms of a geometric series but some used the wrong power in the formula.

Question 7

In part (a), many students assumed incorrectly that mass being proportional to the cube of height meant that $m = kh2$ or even $m = kh$. Part (b) caused problems for many students, unable to deal with the powers involved in the calculation.

Question 8

Most students identified B and C as vertices of odd degree. Part (b) was found to be difficult for the majority of students although some gave one of the two required expressions. The impression gained from many responses was that the students failed to realize that this question was based on the Chinese Postman problem.

Question 9

This was a straightforward question on integration, but many students failed to spot that it integrated to a log function. Students at this level should be able to integrate this function by inspection. Some students lost a mark by not including an arbitrary constant in their solution.

Question 10

Most students realized that the line OB was the radius and that its length was $\sqrt{91}$. Many also realized that angle $\mathrm{B\hat{O}Q}$ was equal to $\frac{\pi}{2} + \mathrm{B\hat{O}A}$. However, a fairly common error was to assume, without any calculation, that $\mathrm{B\hat{O}A} = \frac{\pi}{4}$ instead of $\arctan\left(\frac{8}{5}\right)$. Many students realized that the length of arc BQ is $r\theta$ where r, θ are chosen appropriately, but many failed to carry out this fairly straightforward operation correctly.

Question 11

In part (a), many students chose the correct form of matrix, but it was quite common to see the elements of $R(2\alpha)$ as $2\sin\alpha, \cos 2\alpha$ instead of $\sin 2\alpha, \cos 2\alpha$. In part (b), the matrix multiplication $R(\alpha) \times R(\alpha)$ was often badly done which made part (c) inaccessible to many students. Most of the students who did part (b) correctly, failed to realize that applying $R(\alpha)$ twice was identical to applying $R(2\alpha)$ once.

Question 12

Part (a) was done correctly by the majority of students, but most students were unable to calculate the residual sum of squares in (b). Students are perhaps used to reading the sum of squared residuals on their GDC for standard models but unable to calculate the sum of squared residuals manually for an unusual model.

Question 13

Most students found the modulus and argument successfully in part (a). Most students did this by calculating modulus $= \sqrt{4^2 + 5^2}$ and $\arg = \arctan\left(\frac{5}{4}\right)$ but some student used the mode function on their GDC to do the conversion from Cartesian to polar. Many students were unable to find the area in (b). Some students realized that the formula $0.5bc\sin A$ would be helpful but many were unable to find the correct angle. A minority of students attempted to use the vector method for finding the area of a triangle, but this was almost always unsuccessful.

Question 14

Most students found it difficult to translate the information into mathematics. Many students realized that the expression for X_B would include 2 and $(t-3)$ somewhere but most gave an incorrect expression.

Question 15

Most students made no attempt at this question, suggesting perhaps that the topic is not taught in many schools.

Question 16

The method for solving this question is to put the slope of the ladder $(-\tan 75°)$ equal to the slope of the building $(-5.5\sin 1.1x)$. Many students realized this but, in most cases, the slopes were incorrectly calculated. Even when students reached the correct equation $5.5(\sin 1.1x) = \tan 75°$, some were unable to solve this equation.

Question 17

Parts (a) and (b) were done correctly by many students. Part (c), however, was attempted by only a minority of the students and in almost every case, the necessary algebra proved to be too difficult.

Recommendations and guidance for the teaching of future students

When preparing students for an examination, it is important that the whole syllabus is covered. The impression gained from this year's examination is that, for example, the topics of phase portraits, Markov chains, matrix algebra and indefinite integration were not covered in some schools.

Some students are still losing marks unnecessarily by not giving answers exactly or correct to three significant figures (unless otherwise directed), as instructed on the examination cover.

Students appear to require help in using the Euler method for finding approximate solutions to differential equations. They are aware that the general expression for the iterative formula is given in the formula booklet, but they find this difficult to apply in a particular case.

Higher level paper two

General comments

Some good coverage of the syllabus was noted, and students seemed particularly well prepared with the statistics elements of the course. Many students lost marks through a lack of orderliness in their work, and also from efficient use of their calculators. The Mathematics: A&I course has a lesser emphasis on algebra, but it was noticeable that the students were lacking in the necessary algebraic skills. This is still a higher level Mathematics course and those basic algebraic skills cannot be ignored.

The areas of the programme and examination which appeared difficult for the students

Students were poor with the laying out of their work in an orderly fashion and showing necessary steps. This frequently led to unnecessary errors and to failing to obtain partial credit where examiners suspected that correct methods were applied but the evidence to award marks was not present. Algebraic skills were often lacking.

The areas of the programme and examination in which students appeared well prepared

Students were usually well prepared in most of the statistical work, and the χ^2 question was particularly thorough. The discrete mathematics question showed strong preparation in the topic, although many had difficulties interpreting the final parts of the question, as might be expected in the last question. Students seemed better prepared than previously at continuing with a question even though an earlier part had presented difficulties. Although some students struggled with their calculator skills, there was a general improvement in this respect, with more students showing stronger skills.

The strengths and weaknesses of the students in the treatment of individual questions

Question 1

It was quite shocking to see the number of students that treated parts (a) and (b) as a discrete distribution, leading to no marks in those parts. There were a significant number that gave 2sf answers and even 1sf answers to these parts with the resultant loss of marks. Part (c) was generally much more successfully attempted. Part (d) was the most difficult part of the question, but there was a pleasingly high proportion of students achieving full marks on this. Part (e) was answered by many students as $2x + 4.5$, but the remaining parts were well answered by the majority of the students.

Question 2

There were many good solutions to this entire question. Parts (a) and (b) were generally well answered, with marks being lost with rounding errors. In addition, poor calculator use resulted in incorrect answers with the consequent loss of marks. In part (c) many students did not apply the sine rule correctly, not realizing that they needed to find the third angle. Those that recognized the need to find that angle answered the question well. Again, in this final part, poor calculator use gave rise to a loss of marks in some cases.

Question 3

Parts (a) and (b) were generally well answered. Part (c)(i) was very poorly attempted by many students, although there were many good solutions to the rest of the question. It seems that students are not prepared for a "show that" question where their steps need to be clearly explained. Many students successfully differentiated the expression, although a surprising number of students treated it as two separate applications of the quotient rule, increasing the chances of errors. Many students could successfully find where that derivative equals zero, by a variety of different means. Many students struggled to recognize which equation to substitute into and there were many calculation errors in part (iv).

Question 4

This was generally a high scoring question. Part (a) was well answered, even by the weakest students. A minority of students failed in part (b) which effectively made the rest of the question practically impossible to pick up many marks. It was surprising to see how many students gave the solution as $0.1 \times 0.3 = 0.3$. There were a great many good solutions to part (c). Clearly this is an area in which the students had been well prepared.

Question 5

This question was frequently left blank or poorly attempted. It is clear that many students were unfamiliar with the material. Part (a) was not well understood by many students, although those that were familiar with the concepts found it quite a straightforward question. For part (b), many students tried to solve by separating the variables, but poor algebraic skills let them down. Many found the integral of e^{2y} to be $2e^{2y}$ and many more did not include a "$+c$". Parts (c) and (d) required careful reasoning and it was clear that many students had not been well prepared for these skills.

Question 6

There were a good number of students answering part (a)(i) correctly and slightly fewer answering part (a)(ii) successfully. Part (b)(i) was well answered, but part (b)(ii) eluded the majority of students. Many students were able to answer part (c), but there were few successful attempts at part (d). It seemed that very few students were aware of the central limit theorem applied in this way.

Question 7

For part (a), few students appreciated that they were looking for a volume of a partially filled container, instead attempting to find the volume of the whole container. As a consequence, they were not finding the volume as a function of h. Many students did not understand what the explicit formula was, and left their answer as a definite integral, obtaining only partial marks. Many attempted to rotate about the x-axis instead of the y-axis. There were a number of reasonable attempts at parts (b), and some students were able to obtain marks here from their work in part (a). There were also good attempts for part (c) amongst those who made a reasonable attempt at part (a).

Question 8

Generally, many students obtained good marks on this question. Parts (a) and (b) were well answered. Part (c)(i) was surprisingly poorly answered and although many students were able to obtain the correct answer to part (c)(ii) using the transition matrix; this could not be awarded any marks since the question specifically stated to use their answer to part (b). In part (d) there were few fully correct answers, and these were followed through to some successful solutions for part (e). Frequently calculator skills let students down in finding the correct answer at this point. In part (f), few students appreciated that AC meant that B was not being used, which was not helped by students not labelling their matrix satisfactorily.

Recommendations and guidance for the teaching of future students

Clearly, as always, the full curriculum needs to be covered in order for a student to be successful. It was evident that some students were strong in calculus and weak in statistics, or vice versa. Similarly, it was evident that many students were well prepared for the SL material, but large parts of the additional HL material had been left aside. Calculator skills need to be developed, in particular using the full accuracy of the calculator rather than rounded values. Students should understand that variables can be saved and used for subsequent parts of the question.

Higher level paper three

General comments

In general students found the paper accessible though there were some indications that the pressure of time had a detrimental effect in the final parts for a few students.

The first question was based on the statistics component of the course and consisted of three different tests from the syllabus. The χ^2 test for independence was well done, but the test for the binomial parameter and the two-sample *t*-test were less successful.

The second question began with an arithmetic sequence but largely involved the kinematics section of the course and in particular projectile motion. Many students were clearly familiar with the topic and managed to produce some good answers.

As always, student ability to access the questions depends on having been taught the course, and there are signs that some schools are entering students who, for example, have not covered much of the statistics topic.

The areas of the programme and examination which appeared difficult for the students

- Recognizing which statistical test to use
- The test for a binomial parameter
- Working with parameters rather than numbers when modelling
- Properties of a quadratic curve

The areas of the programme and examination in which students appeared well prepared

- Conditional probabilities from a two-way table
- The χ^2 test for independence
- Arithmetic sequences
- Use of the calculator to find features of a curve

The strengths and weaknesses of the students in the treatment of individual questions

Question 1

Parts (a), (b) and (d) involved finding information from a two-way table and were completed correctly by most students.

Part (c) was a straightforward χ^2 test for independence. Some students had clearly not been taught how to do it but those who had were generally successful in completing the question. It is important when stating the null and alternative hypotheses that the two variables are identified.

In part (e), students need to be aware that when using a binomial distribution the assumptions are: there are two outcomes (with constant probabilities for each trial) and the trials are independent of each other. Either of these would have received the mark.

Part (f) tried to make it clear which test should be done. The hypotheses were given and referred to the proportion developing cracks, and it was stated a binomial distribution was being assumed. Many students though did not recognize the test, even though it is in the syllabus, and others tried to perform a χ^2 test in order to test the distribution.

One of the main problems encountered was finding the observed number of components that developed cracks and then using this for their test. A commonly seen error was the use of the probability density function, rather than the cumulative density function, which indicates a lack of understanding of critical regions.

In part (g), there were a few possible answers to these questions and most students managed to suggest a valid reason for preferring at least one of the tests. Some answers though, such as 'the second test did not use all the data', showed a lack of understanding on the part of the student.

Part (h) was intended to be a typical two sample t-test. The instruction was simply to 'perform' the test. The test is incomplete unless the student formally stating the hypotheses and this was often not done; these problems might be set as discrete sub-parts, each asking for stage in the test, or (as here) they might be asked in a single question, *but* all elements should be included in the response. There were some who stated the hypotheses in the form of the sample mean, $H_0\colon \bar{t}_1 = \bar{t}_2$ which again shows a lack of understanding. The hypotheses could be stated in words. When stating in words though it needs to be made clear that the population means and not the sample means are being referred to. It is probably easiest for these tests to use μ in the hypotheses, though if using μ_1 and μ_2 it is important to say which is which.

The test itself was usually carried out correctly. On this occasion the students could use either pooled or non-pooled values for the standard deviation as there were no instructions in the question regarding which one was preferred.

Part (i) was designed to test the understanding of the difference between a 'statistically significant' result and one that is 'significant' in context.

Question 2

Part (a) involved the use of arithmetic sequences. It was generally well done.

In part (b), students were not able to make a start on the question. Possibly due to having not covered the work, though some might have been put off by having an unknown parameter in the model. Working with a variable and a parameter represented by a letter is a common feature of modelling, and one that will need practice to gain familiarity.

In part (b)(ii), many students substituted $t = 3$. In a 'show that' question the given answer cannot form part of the working. That technique is more appropriate for the "verify" command.

In part (c), most students recognized the values they needed to substitute into the given height expression.

Part (d) was a more difficult 'problem solving' question. There were several steps to be completed, and a value of t to be found in terms of θ and then substituted. It was pleasing to see this done successfully by some students. Other students demonstrated good examination technique by taking the given equation and finding its maximum for part (d)(ii). Some were clearly aware that the maximum range occurs when the angle of projection is $45°$ and of course this was also a valid approach.

Part (e) involved finding the equation of a parabola from a maximum point and one other point, and using knowledge of the features of a quadratic. The value of c being the maximum was stated by most. The value of $b = 0$ could be obtained from prior knowledge, or derived from the formula $x = -\frac{b}{2a}$ or in a variety of other ways, many of which were seen in students answers. Most who found these first two were able then to use the second point to find the value of a.

The first part of answering part (f), finding the coordinates of the seat, was well done by many students who were able to successfully link the early and late parts of the question. The final part was not attempted by many, in some cases perhaps due to a lack of time and in others because they had not successfully found an equation in the previous part.

Recommendations and guidance for the teaching of future students

When teaching the statistical tests, it is important to have a good understanding of the general theory, the meaning of a critical region, *p*-value or significance level. The test for a binomial parameter is one that does need careful teaching, but the effort is worthwhile because, unlike those that can be done directly on a GDC, this requires a student to understand the concepts mentioned above.

In modelling, students should become familiar with working with equations that contain unknown parameters, and thereby finding expressions for significant features rather than values.